Contribution to Regularization in Optimal Trajectory Problems

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Optimizing the trajectory of a low thrust space vehicle usually means solving a nonlinear two-point boundary value problem. In general, accuracy requirements necessitate extensive computation times. In celestial mechanics regularizing transformations of the equations of motion are used to eliminate computational and analytical problems that occur during close approaches to gravitational force centers. It was shown in previous investigations that regularization in the formulation of the trajectory optimization problem may reduce the computation time. In this study, a set of regularized equations describing the optimal trajectory of a continuously thrusting space vehicle is derived. The computational characteristics of the set are investigated and compared to the classical Newtonian unregularized set of equations. The comparison is made for low thrust minimum time escape trajectories. The comparison indicates that in the cases investigated for bad initial guesses of the known boundary values a remarkable reduction in the computation time was achieved. Furthermore, the investigated set of regularized equations shows high numerical stability even for long duration flights and is less sensitive to errors in the guesses of the unknown boundary values.

Introduction

THE space vehicle trajectory optimization problem, in general, is treated from the point of view of the Mayer-Bolza problem which is well known in the calculus of variations.[1,2] Usually the solution must be obtained numerically requiring some means of iteration procedures involving multiple integrations of sets of differential equations. One of the primary considerations in evaluating a numerical optimization procedure is the computation time for a predetermined accuracy with which the terminal conditions have to be satisfied. It is known that, when space vehicle trajectories cross regions with strongly varying gravitational force fields, the necessary numerical accuracy often requires extreme computation times. The computation time depends in principal on: 1) the numerical integration procedure, 2) the iterative numerical optimization method, and 3) the mathematical formulation of the problem used.

In this study, standard integration[3] and standard iteration[4,5] techniques are used. The main purpose of the investigation was to develop an improved formulation of the set of differential equations describing the space vehicle motion and the optimality conditions; in other words, to find a set of equations with high stability of numerical integration and less sensitivity with respect to errors in the required guesses for the unknown boundary conditions. This behavior is necessary for the iteration procedure to demonstrate good convergence characteristics in isolating the optimal solution.

In celestial mechanics, regularizing transformations are used to remove the singularity which occurs due to the r^{-2} term in the equations of motion during close approach to a gravitational force center. These techniques reduce or eliminate computational and/or analytical problems in calculating those parts of space object trajectories.[6] It was shown in previous investigations[7-9,19] that the use of regularizing transformations in the formulation of the trajectory optimization problem may reduce the computation time and improve the convergence characteristics in comparison with unregularized sets of equations. Based on those results, in this study the equations for the optimal trajectory of a space vehicle with a continuous low thrust propulsion system are derived using regularized variables. The set of variables chosen was first described by Sperling[11] and Burdet.[12] The regularization for the state, as well as for the Lagrangian multiplier, equations is obtained by using only the classical Sundman[10] time transformation.

To investigate the numerical behavior of the derived system, the examples chosen are two-dimensional Earth minimum time escape trajectories starting from various orbits and using various vehicle characteristics. The results are compared with the classical unregularized set of Newtonian equations of motion.[6,8,16,18]

The comparison indicates that in the cases investigated, a reduction in computation time was achieved and that the regularized set of the equations of motion is less sensitive to errors in the guesses of the unknown boundary conditions.

Problem Formulation

The equation of motion of a spacecraft with a continuous thrusting propulsion system in an inverse-square gravitational force field can be written as

$$\ddot{\bar{r}} + \mu(\bar{r}/r^3) = \bar{P} \tag{1}$$

where the superscribed dot denotes the derivative with respect to t, $(\bar{\ })$ indicates the quantity as a vector, $\bar{r}$ is the radius vector from the force center to the vehicle, which is assumed to be a point mass, r is the absolute value of the radius vector, $\mu = GM$, where G is the universal gravitational constant, M is the mass of the central body, and $\bar{P}$ is the vector of the thrusting acceleration defined as

$$\bar{P} = (F/m)\bar{u} \tag{2}$$

with F, the absolute value of the thrust, which is assumed to be constant in this study, $m = m_0 - \beta t$ is the vehicle mass at time t with β as the constant mass flow rate. $\bar{u}$ is the unit vector of the unspecified thrust direction.

The trajectory has to connect the given initial state

$$\bar{r}_0 = \bar{r}(t_0) \qquad \dot{\bar{r}}_0 = \dot{\bar{r}}(t_0) \qquad m_0 = m(t_0) \tag{3}$$

and the final state

$$\bar{r}_f = \bar{r}(t_f) \qquad \dot{\bar{r}}_f = \dot{\bar{r}}(t_f) \tag{4}$$

The final time is either predetermined or free.

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The optimal transfer problem considered can then be formulated as follows: determine the unspecified thrust history $\bar{u}$ to minimize any performance index J, a function of the boundary values of the state $\bar{x}$ where

$$\bar{x}^T = (\bar{r}, \dot{\bar{r}}, m)$$

and the time t

$$J = J(\bar{x}_0, t_0, \bar{x}_f, t_f) \tag{5}$$

while satisfying the differential equations

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}, t) \tag{6}$$

for $t_0 \leqq t \leqq t_f$ and the boundary conditions given in Eqs. (3) and (4). The derivation of the set of necessary conditions for an optimal trajectory previously formulated as a Mayer-Bolza problem may be found in any textbook of the calculus of variations.[13]

Derivation of the Regularized Equations

The derivation of regularized differential equations for the state variables and the Lagrangian multipliers can be approached either by first regularizing the equations of motion and then setting down the necessary criteria for optimization in terms of the new independent variable, or by setting up the complete optimization problem with ordinary time as the independent variable and then applying the regularizing transformation to the whole system of resulting equations. In the following derivation, the first approach is used. The second approach leads to the same set of equations if the second-order equation of motion is reduced to a set of first-order equations considering the fact that the regularized velocity using the time transformation $dt = r\,ds$ will be $\bar{v} = r\dot{\bar{r}}$.

Before applying the new independent variables, defined by the classical regularizing transformation of Sundman[10]

$$ds = r^{-1}\,dt \tag{7}$$

to the equations of motion, we may write Eq. (1) as

$$\begin{aligned} r^2\ddot{\bar{r}} &= -\bar{r}(\dot{\bar{r}}\cdot\dot{\bar{r}}) + 2h\bar{r} - \mu\bar{e} + r^2\bar{P} \\ \dot{h} &= (\dot{\bar{r}}\cdot\bar{P}) \\ \dot{\bar{e}} &= (1/\mu)[\dot{\bar{r}} \times (\bar{r} \times \bar{P}) + \bar{P} \times (\bar{r} \times \dot{\bar{r}})] \end{aligned} \tag{8}$$

where the following substitution equations for energy h and the Laplacian constant $\bar{e}$ and their time derivatives were used

$$h = \frac{(\dot{\bar{r}}\cdot\dot{\bar{r}})}{2} - \frac{\mu}{r}$$

and the Laplacian constant $\bar{e}$

$$\bar{e} = -\frac{\bar{r}}{r} + \frac{1}{\mu}[\dot{\bar{r}} \times (\bar{r} \times \dot{\bar{r}})] \tag{10}$$

Because of Eq. (7) the second time derivative of the radius vector is

$$r^2\ddot{\bar{r}} = \bar{r}'' - [(\bar{r}\cdot\bar{r}')/r^2]\bar{r}' \tag{11}$$

The prime denotes the derivative with respect to the new independent variable s.

Equations (7, 8, and 11) together with the differential equation for the mass variation $\dot{m} = \beta$ give the complete set of regularized differential equations for the state variables. Using the substitution $\bar{r}' = \bar{v}$ the equations can be written in a first-order form[12]

$$\begin{aligned} \bar{r}' &= \bar{v} \\ \bar{v}' &= 2h\bar{r} - \mu\bar{e} + r^2\bar{P} \\ h' &= (\bar{v}\cdot\bar{P}) \\ \bar{e}' &= (1/\mu)[\bar{v} \times (\bar{r} \times \bar{P}) + \bar{P} \times (\bar{r} \times \bar{v})] \\ m' &= \beta r \\ t' &= r \end{aligned} \tag{12}$$

Setting up the Hamiltonian of system (12) and deriving its partial derivatives with respect to all of the state variables will lead to the following set of Lagrangian multiplier equations, the necessary conditions of optimality

$$\begin{aligned} \bar{\lambda}_r' &= -\bar{\lambda}_v 2h - (\bar{\lambda}_v\cdot\bar{P})2\bar{r} - [\bar{v} \times (\bar{\lambda}_e \times \bar{P}) + \bar{P} \times \\ &\quad (\bar{\lambda}_e \times \bar{v})]/\mu - \lambda_m\beta(\bar{r}/r) - \lambda_t(\bar{r}/r) \\ \bar{\lambda}_v' &= -\bar{\lambda}_r - \lambda_h\bar{P} + [\bar{r} \times (\bar{\lambda}_e \times \bar{P}) + \bar{\lambda}_e \times (\bar{r} \times \bar{P})]/\mu \\ \lambda_h' &= -(\bar{\lambda}_v\cdot\bar{r})2 \\ \bar{\lambda}_e' &= \bar{\lambda}_v\mu \\ \lambda_m' &= -(F/m^2)|\bar{H}_p| \\ \lambda_t' &= 0 \end{aligned} \tag{13}$$

The subscript denotes to which of the state variables the multiplier λ_i is related.

$|\bar{H}_p|$ is the absolute value of the partial derivative of the Hamiltonian H with respect to the thrusting acceleration $\bar{P}$

$$\bar{H}_p = r^2\bar{\lambda}_v + \lambda_h\bar{v} + (1/\mu)[\bar{r} \times (\bar{v} \times \bar{\lambda}_e) + \bar{\lambda}_e \times (\bar{v} \times \bar{r})] \tag{14}$$

The optimal thrust vector program is derived from the condition that the partial derivative of the Hamiltonian with respect to the control $\bar{u}$, which obeys the constraint equation

$$(\bar{u}\cdot\bar{u}) - 1 = 0 \tag{15}$$

has to be zero.

It follows

$$\bar{u} = -\bar{H}_p/|\bar{H}_p| \tag{16}$$

This sign in Eq. (16) follows from examining the second variation of the Hamiltonian with respect to $\bar{u}$ for minimization.

Discussion

Equations (2, 12–14, and 16) give the complete set of regular equations for an optimal trajectory of a thrusting space vehicle in an inverse square force field.

The derived system is seen to be regular without any limit value considerations except for the unit vector of the radius and without a transformation of the state variables. Although the unit vector of the radius has for $r \to 0$ a well known limit value,[15] in rectangular coordinates the calculation of $\bar{r}/r$ near a gravitational singularity will lead to numerical difficulties. In the case of calculating a two-dimensional trajectory, the use of polar coordinates will prevent those problems. The increase of the number of differential equations is a minor disadvantage with respect to the integration time. Compared to the classical formulation of the trajectory optimization problem[2] in the three-dimensional case, the number of equations increases from 7 to 12 for the state description and from 7 to 11 for the adjoint system. Those additional equations are one vector equations for the Laplacian constant and two scalar equations for the energy and the time, which is now a new dependent variable. The adjoint system increases one equation less than the state system because the multiplier related to the time t is a constant, as is pointed out in Appendix A. If the time would appear explicitly in any of the Eqs. (12) and (13), it always could be eliminated by using Eq. (7).

The integration interval changes from $t_0 \leqq t \leqq t_f$ to $s_0 \leqq s \leqq s_f$, with s_f unknown even in the case that t_f is given, because s_f depends on the unknown history of the radius of the optimal trajectory.[14]

In contrast to the classical set of equations for the optimal trajectory, the equation of the Lagrangian multiplier associated with the differential equation of the vehicle mass λ_m is not independent.

For two-dimensional trajectory calculations, the use of polar coordinates may be advantageous, because the number of differential equations decreases to 7 for the state description and to 5 for the adjoint system. The Laplacian constant does not appear in the equations of motion. It is well known that the regularizing transformation used will not regularize the differential equations for the central angle. The adjoint system has one equation less due to the independence of the central angle equation from the optimization problem.

Discussion of Numerical Results

The numerical behavior of the system derived was investigated by means of the two-dimensional minimum time Earth escape. For the minimum time Earth escape of the continuous thrusting vehicle with $\beta = $ const mass flow rate, the performance index was formulated as

$$J = (-m_f) \tag{17}$$

The only boundary condition at the final time is

$$h = 0 \tag{18}$$

Therefore, an auxilliary function ϕ is constructed

$$\phi = -m_f + vh \tag{19}$$

(where v is a constant Lagrangian multiplier). The final values of the Lagrangian multipliers at the final time t_f with the definition $\lambda_i(t_f) = \lambda_{if}$ are

$$\lambda_{if} = 0$$
$$\lambda_{mf} = -1 \tag{20}$$
$$\lambda_{hf} = v$$

The first equation of set (20) indicates that all final λ values, except λ_{mf} and λ_{hf} are zero.

v may be expressed in terms of the unknown state variables of the final time t_f using the condition that the Hamiltonian has to be equal to zero even at the final time. Using this condition and applying Eqs. (20) gives

$$v = -|\beta| r_f m_f / F |v_f| \tag{21}$$

Equations (2, 12–14, and 16) together with the boundary conditions (3, 18, 20, and 21) form the complete set which is to be solved for the optimum escape trajectory calculation. The reason that the optimal trajectory is not determined by one integration of this set is that the state variables and multipliers are not given as a complete set at one definite time t. So the solution is an iteration process starting from any initial guess of the missing boundary values at the initial or final time. Usually, an initial guess of the Lagrangian multipliers is made.

In this study approximations of the initial values of the multipliers were determined from a backward integration of an escape trajectory using a tangential thrust program. This method made use of the knowledge that for escape, a tangential thrust

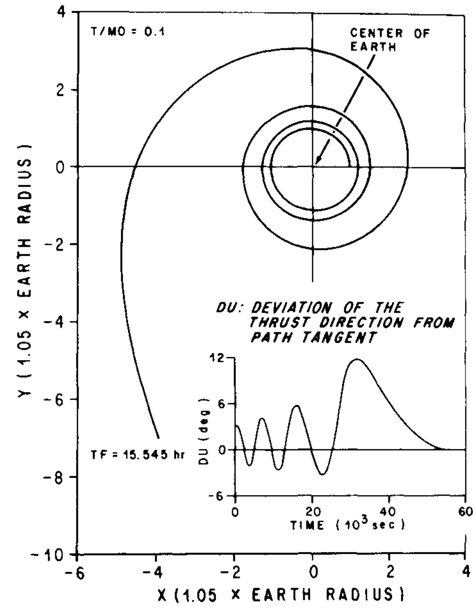

Fig. 1 Optimal low thrust Earth escape spiral.

Table 1 Numerically integration characteristics for different types of optimal Earth escape trajectories

TRAJECTORY TYPE	SET OF EQUATIONS	COORDINATE SYSTEM	MAX ERROR PER STEP	INTEGR. TIME PER STEP	NO. OF STEPS	ERROR IN H** t_i	ERROR IN H** t_f	QUADRATIC*** ERROR
SPIRAL	NEWTONIAN	RECTANGULAR	10^{-5}	100%	52	$< 10^{-7}$	$< 10^{-8}$	$< 10^{-12}$
~ 3.7 REV.	REGULARIZED	RECTANGULAR	10^{-5}	~ 200%	37	$< 10^{-8}$	$< 10^{-7}$	$< 10^{-13}$
t_f = 15.45 hr	REGULARIZED	POLAR	10^{-4}	~ 100%	24	$< 10^{-7}$	$< 10^{-9}$	$< 10^{-14}$
SPIRAL ~ 240 REV. t_f = 35 Days	REGULARIZED	POLAR	10^{-4}	~ 100%	989	$< 10^{-6}$	$< 10^{-7}$	$< 10^{-12}$
SPIRAL ~ 700 REV. t_f = 165 Days	REGULARIZED	POLAR	10^{-4}	~ 100%	--	--	$< 10^{-7}$	$< 10^{-12}$
HIGHLY ECC.	NEWTONIAN	RECTANGULAR	10^{-5}	100%	26	$< 10^{-6}$	$< 10^{-7}$	$< 10^{-13}$
t_f = 111 Hr	REGULARIZED	RECTANGULAR	10^{-5}	~ 200%	14	$< 10^{-6}$	$< 10^{-6}$	$< 10^{-12}$
HIGHLY ECC.* t_f = 56 Days	REGULARIZED	RECTANGULAR	10^{-4}	~ 100%	132	$< 10^{-5}$	$< 10^{-8}$	$< 10^{-13}$

* INTEGRATION WITH RUNGE KUTTA 4(5), ALL OTHER CASES WITH RUNGE KUTTA 7(8) ON MSFC'S SDS 930 COMPUTER.

** HAMILTONIAN OF THE SYSTEM CONSIDERED.

*** SUM OF THE QUADRATIC ERRORS IN THE BOUNDARY CONDITIONS AT THE TIME T_F.

Table 2 Convergence characteristics of different kinds of Earth escape trajectories κ_i calculated by a tangential thrust backward integration

TYPE OF ESCAPE	SET OF EQUATIONS	COORDINATE SYSTEM	MAX ERROR PER STEP	QUADRATIC ERROR	NO. OF ITERATIONS	NO. OF TRAJECTORIES CALCULATED
SPIRAL 15.45 HR.	NEWTONIAN	RECTANGULAR	10^{-5}	$< 10^{-12}$	19	26
	REGULARIZED	RECTANGULAR	10^{-4}	$< 10^{-12}$	6	16
	REGULARIZED	POLAR	10^{-4}	$< 10^{-13}$	4	11
SPRIAL 35 DAYS	REGULARIZED	POLAR	10^{-4}	$< 10^{-12}$	30	37
SPIRAL 165 DAYS	REGULARIZED	POLAR	10^{-4}	$< 10^{-12}$	18	25
HIGHLY ECC. 111 HR.	NEWTONIAN	RECTANGULAR	10^{-5}	$< 10^{-12}$	29	36
	REGULARIZED	RECTANGULAR	10^{-4}	$< 10^{-15}$	6	16
HIGHLY ECC. 56 DAYS	REGULARIZED	RECTANGULAR	10^{-4}	$< 10^{-13}$	13	23

program is close to the optimal steering program. For the integrations, standard Runge-Kutta-Fehlberg[3] procedures of different orders as indicated in Table 1 were used.

The first example calculated is a 15.5 hr Earth escape spiral from a low circular Earth orbit with $r = 1.05$ times Earth radius and an initial thrust to mass ratio of 0.1 m/sec^2. This example was also investigated in Ref. 18. Although this F/m_0 is relatively large, it represents a compromise between a computationally expensive realistic trajectory and an inexpensive unrealistic trajectory. The optimal trajectory is shown in Fig. 1 with the optimal thrusting angle measured against the path tangent.

The second example investigated is shown in Fig. 2. It is an optimal minimum time Earth escape starting from a highly eccentric Earth orbit near the Earth with $r_0 = 1.34 \times$ Earth radius. The gravitational force varies until the final time is reached by approximately a factor of 10^3 in magnitude, thus, it is expected that for certain error bounds, the numerical integration stepsize will change within a wide range. The initial thrust to mass ratio of 3.10^{-3} m/sec^2 is somewhat more realistic than is the spiral escape. Figure 2 shows the optimal steering program. The spacecraft in both cases is considered as a point mass with

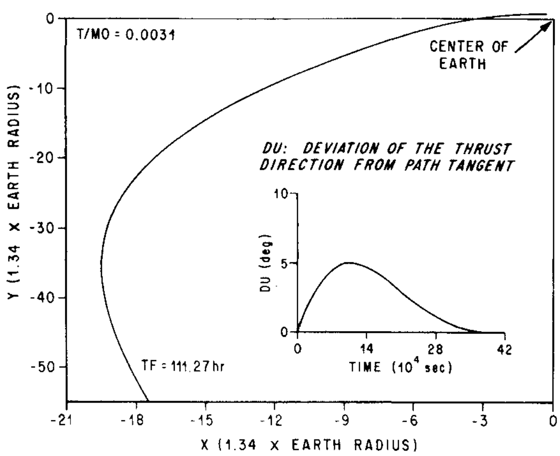

Fig. 2 Optimal low thrust Earth escape starting from a highly eccentric orbit.

a continuous thrusting device. The mass flow rate is constant. All examples are calculated in two dimensions.

Table 1 gives some insight into the integration characteristics of the regularized set derived in comparison to the Newtonian set of equations of motion, which was also investigated in a first-order form. Some results, not shown in Table 1, indicate that the use of a second-order formulation of the Newtonian set of equations may save some integration time. It did not improve the convergence behavior of the system in those cases investigated.

The examples for the Newtonian set of equations were calculated in a rectangular coordinate system. For the regularized set of equations, rectangular and polar coordinates were used.

The integration time per step is given only in normalized form for comparison with the Newtonian case, which is chosen as 100%. The actual values for the SCS-930 used for the calculations are not representative of modern, faster computers (e.g., UNIVAC 1108). The values given are based on a rough calculation from the duration of one complete trajectory integration divided by the number of steps. The number of stepsize changes is not considered, but means usually each time one additional step.

The doubling in the integration time per step for the regularized set in rectangular coordinates in comparison to the Newtonian set is essential due to the increased number of differential equations. That also explains the approximate equality of the integration time in the polar coordinates where the number of equations is equal to that of the Newtonian formulation.

The number of steps for the same error bound required in the boundary conditions at the final time is, in the 15 hr spiral case, decreased by approximately 30% and in the 111 hr escape case more than 40% as compared with the Newtonian set and the regularized set calculated in rectangular coordinates. The use of polar coordinates for this particular example seems advantageous even in the number of steps.

Since a closed form solution to the problem considered here does not exist, the error generated by the numerical integration process is unknown. The converged trajectory is always chosen as the optimal trajectory.

The error in the Hamiltonian given in Table 1 indicates, that all calculations are made within the same error limits and gives an impression of the numerical accuracy achieved.

Table 3 Initial error influence on the convergence characteristics of the 15.45 H Earth escape spiral

$\alpha = \frac{\lambda_i}{\lambda_{OPT}} \left[\frac{\text{initial } \lambda}{\text{optimal } \lambda}\right]$	NEWTONIAN SET OF EQUATIONS IN RECTANGULAR COORDINATES		REGULARIZED SET IN RECTANGULAR COORDINATES	
	N*	n**	N	n
.01	Not Conv.***	- - -	Not Conv.	- - -
.1	Not Conv.	- - -	21	31
.5	17	24	8	18
.8	7	14	5	15
1.0	0	1	0	1
1.2	7	14	6	16
2.0	16	23	16	26
10.0	19	26	20	30
100.0	Not Conv.	- - -	30	40

N * - NUMBER OF ITERATIONS

n ** - NUMBER OF TRAJECTORIES CALCULATED TO FIND THE OPTIMAL SOLUTION.

*** - NOT CONVERGED WITHIN 50 ITERATIONS

A certain control of the results is possible by comparing the final time with time necessary to achieve zero energy with a tangential thrust steering program. For each problem considered, the isolated trajectories calculated with different sets of equations and coordinate systems coincide at least within the number of digits indicated by the maximum allowable error per integration step.

The quadratic error given in the last column of Table 1 is defined as the sum of the quadratic error of the boundary conditions at the final time. The routine requires only that the value be smaller than 10^{-12}. The extent to which this criterion is exceeded is not controlled.

To demonstrate the numerical stability of the derived regularized set, three long duration optimal escape solutions are shown in Table 1. The 30-day spiral escape trajectory starts from a 1.05 Earth radius circular orbit, with an initial thrust to mass ratio from 10^{-3} m/sec^2. The number of revolutions is more than 235. The 165-day spiral escape starts from the same circular Earth orbit with an initial thrust to mass ratio of $5 \cdot 10^{-4}$ m/sec^2. The number of revolutions is approximately 700. The 56-day escape trajectory starts from the high eccentric orbit.

Table 2 shows the number of iterations and the number of trajectories calculated to isolate the optimal trajectory for all those cases listed in Table 1. The initial guesses of the unknown Lagrangian multipliers were in all cases systematically derived

Table 4 Initial error influence on the convergence characteristics of the 111 H Earth escape trajectory

$\alpha = \frac{\lambda_i}{\lambda_{OPT}} \left[\frac{\text{initial } \lambda}{\text{optimal } \lambda}\right]$	NEWTONIAN SET OF EQUATIONS IN RECTANGULAR COORDINATES		REGULARIZED SET IN RECTANGULAR COORDINATES	
	N*	n**	N	n
.1	Not Conv.	- - -	20	30
.5	35	42	10	20
.8	38	45	--***	---
1.0	0	1	0	1
1.2	20	27	---	---
1.5	38	45	8	18
10.0	69	76	20	30

N * - NUMBER OF ITERATIONS

n ** - NUMBER OF TRAJECTORIES CALCULATED TO FIND THE OPTIMAL SOLUTION

*** - NOT CALCULATED

from a backward calculation starting from the final state, which was obtained from a tangential thrust steering program. The deviation of those λ values from the finally calculated optimal values differ depending on the individual sensitivity of those multipliers with respect to a nonoptimal steering program. The deviations range from 10% to about an order of magnitude and, most importantly, include changes in signs.

Table 2 shows the improved behavior of the regularized set in performing those calculations. The figures given are possibly not the best values attainable. The influence of different scaling systems or optimal selections of sensitivity factors for the optimization technique used is not considered for this comparison. The long duration cases were able to be calculated only with the aid of the regularized set of equations of motion.

To investigate the sensitivity of the different sets of equations with respect to deviations of the initial λ values from the optimal values in Tables 3 and 4, the results of the nonrealistic systematic changes in the initial λ's are given for the shorter escape cases.

It is found that for small deviations due to the smaller number of differential equations, the Newtonian set may require less trajectory calculations than the regularized set. But for greater deviations, the regularized set is obviously less sensitive. The remark "nonconverged" indicates that after about 50 iterations, no improvement of the isolation procedure was noticed. Also, for the figures in Tables 3 and 4, no effort was made to derive the smallest possible number of iterations. For the sensitivity analysis, the error in the final time was always considered to be zero.

Conclusions

Based on the results shown in this paper, the following conclusions may be drawn.

For the examples calculated, the integration time per trajectory necessary in either Newtonian or regularized formulation of the equations of motion is found to be approximately equal. The gain due to the smaller number of steps using the regularized set is lost due to the greater number of differential equations which cause a higher integration time per step.

In the case of good initial guesses of the missing boundary values (i.e., the Lagrangian multipliers at the time $t = 0$) the smaller number of differential equations in the Newtonian formulation is even advantageous with respect to the numerical optimization method used. Only slight differences between both formulations were found.

On the other hand, a remarkable gain in computation time is achievable for the trajectory types investigated by using the regularized set in the more realistic cases of bad initial guesses for the Lagrangian multipliers. The smaller sensitivity of the regularized set leads more rapidly to the converged trajectory. In the case of extremely bad initial values, the regularized set gives the hope that convergence may be achieved where it otherwise might have been impossible. In the long duration cases presented, convergence was achieved without use of aids other than were previously mentioned and without major difficulties.

Appendix A: An Alternate Way to Derive the Regularized Equations

Using the same substitutions of energy h and Laplacian constant Eqs. (9) and (10) in the equations of motion (1), we find with the additional substitution of the velocity $\bar{v} = \dot{\bar{r}}$ the first-order system of the equations of motion

$$\begin{aligned} \dot{\bar{r}} &= \bar{v} \\ \dot{\bar{v}} &= -(1/r^2)[\bar{v}(\bar{r}\cdot\bar{v}) - 2h\bar{r} + \mu\bar{e} - r^2\bar{P}] \\ \dot{h} &= (\bar{v}\cdot\bar{P}) \\ \dot{\bar{e}} &= (1/\mu)[\bar{v}\times(\bar{r}\times\bar{P}) + \bar{P}\times(\bar{r}\times\bar{v})] \\ \dot{m} &= \beta \end{aligned} \tag{A1}$$

If we start setting up the optimization problem with the system (A1) and using the time transformation $dt = rds$, we will not find the same set of Lagrangian multiplier Eqs. (13). In particular, the system derived in such a manner will be more complex and not regularized, and the Lagrangian multiplier equation related to the differential equation of the vehicle mass will stay independent as in the classical case.

But starting from the second-order equation for the radius vector

$$r^2\ddot{\bar{r}} = -\dot{\bar{r}}(\bar{r}\cdot\dot{\bar{r}}) + 2h\bar{r} - \mu\bar{e} + r^2\bar{P} \tag{A2}$$

and considering the fact that the regularized velocity using the time transformation $dt = rds$ will be

$$\bar{v} = r\dot{\bar{r}} \tag{A3}$$

then the following first-order system will be derived

$$\begin{aligned} \dot{\bar{r}} &= \bar{v}/r \\ \dot{\bar{v}} &= (2h\bar{r} - \mu\bar{e} + r^2\bar{P})/r \\ \dot{h} &= (\bar{v}\cdot\bar{P})/r \\ \dot{\bar{e}} &= \{(1/\mu)[\bar{v}\times(\bar{r}\times\bar{P}) + \bar{P}\times(\bar{r}\times\bar{v})]\}/r \\ \dot{m} &= \beta \end{aligned} \tag{A4}$$

Starting with set (A4) setting up the optimization problem gives the same Lagrangian multiplier equations as in the first approach described previously.

The relation between the Hamiltonian ${}^sH$ of the regularized set of state equations (12) and the Hamiltonian ${}^tH$ of the unregularized set (A4) is

$${}^sH = r\,{}^tH \tag{A5}$$

Building the partial derivatives gives the connection between the regularized and unregularized set of multiplier equations.

$$\begin{aligned} \lambda_r' &= \dot{\lambda}_r r - \frac{\bar{r}}{r}{}^tH = -\frac{\partial}{\partial\bar{r}}(r\,{}^tH) = -\frac{\partial\,{}^sH}{\partial\bar{r}} \\ \lambda_v' &= \dot{\lambda}_v r \\ \lambda_h' &= \dot{\lambda}_h r \\ \lambda_e' &= \dot{\lambda}_e r \\ \lambda_m' &= \dot{\lambda}_m r \end{aligned} \tag{A6}$$

The second part of the first equation in set (A6) appears because of the defining equation of the new independent variable. The additional multiplier λ_t follows for the autonomous system (12) and as pointed out previously, considering the necessary free boundary condition s_f as the constant

$$\lambda_t = -{}^tH \tag{A7}$$

$|{}^tH|$ is the value of the Hamiltonian of the unregularized system (A4).[14]

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Filtering and Smoothing Simulation Results for CIRIS Inertial and Precision Ranging Data

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The CIRIS (Completely Integrated Reference Instrumentation System) under development at Holloman Air Force Base will provide a continuous indication of aircraft position, velocity, and attitude based on the optimal combination of inertial navigation and precision radio range and range-rate data. A postflight data processor, developed by Intermetrics, accomplishes the optimal estimation, either by forward filtering or by smoothing. The simulation results show that the CIRIS accuracy will be a strong function of the placement of the ranging system ground transponders. Optimal smoothing of the data significantly improves the estimation accuracy during the gaps in ranging-system coverage.

Introduction

THE CIRIS (Completely Integrated Reference Instrumentation System) under development at Holloman Air Force Base will provide highly accurate indications of aircraft position, velocity, and attitude. These data are required in connection with the flight testing and evaluation of navigation and guidance systems. The onboard CIRIS subsystems (Fig. 1) include a Litton CAINS inertial navigation system, a Cubic Corp. CR-100 precision range/range-rate measurement set, a Hewlett-Packard HP-2100 general purpose digital computer, and a flight data recorder. The ranging system measures the range and the integral of the range rate to cooperative transponders placed at known surveyed locations. Synchronized position/velocity/attitude data from the inertial system and range/range-rate data from the ranging system are recorded on the flight data tape.

A postflight data processor, developed by Intermetrics, computes optimal estimates of aircraft position, velocity, and attitude, using the inertial and ranging data. At user option, the estimates are computed either by a Kalman forward filter or by an optimal smoother. The Kalman forward filter[1] determines an estimate $\hat{x}_f$ of the state vector based on all the past data leading up to each time point of interest. The optimal smoother utilizes not only all the past data, but all the future data, with respect to each time point of interest. Several alternate mathematically equivalent smoother formulations are available.[2-6] The most simple formulation conceptually is the Fraser two-filter smoother.[5] In the two-filter smoother, one obtains from an optimal forward filter an estimate $\hat{x}_f$ and associated covariance P_f of the estimation error. An optimal backward filter then generates an estimate $\hat{x}_b$ and associated covariance P_b, based on all the future data with respect to each time point of interest. Since statistically independent data have been used to generate the forward estimate

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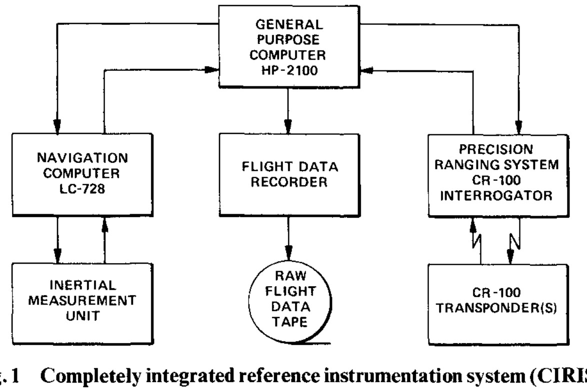

Fig. 1 Completely integrated reference instrumentation system (CIRIS).